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COMPLETENESS AND DECIDABILITY OF THE DEDUCIBILITY PROBLEM FOR SOME CLASS OF FORMULAS OF SET THEORY

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An extension of results on the decidability of classes of formulas in set theory is proved. In particular a class of restricted quantified formulas is proved to be decidable also in the case in which the underlying axiomatic set theory does not contain the axiom of foundation. For all the classes considered is also studied whether or not they result to be not only decidable, but also complete and a simple decidable but not complete class of formulas is presented.

1. Introduction and Motivations.

The present work extends some of the results obtained in *Decision Procedures for Elementary Sublanguages of Set Theory II. Formulas Involving Restricted Quantifiers, together with Ordinal, Integer, Map, and Domain Notions* [1].

The results contained in [1] are based upon a decidability proof of the problem of deducibility in Zermelo-Fraenkel-Skolem set theory

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ZF of a class of formulas T which is then enlarged in several stages.

To say that the *deducibility problem* for a given class of formulas is decidable with respect, say, to ZF means that in a given metatheory (for example again ZF) it is possible to introduce a one-place predicate $T(x)$ that expresses the fact that (the code of) a formula belongs to the class T and another one-place *recursive* predicate $D(x)$ which analogously expresses the fact that:

$$ZF \vdash \forall\varphi(T(\varphi) \rightarrow (D(\varphi) \leftrightarrow ZF \vdash \varphi)) \tag{1}$$

with ZF intended as metatheory and φ as variable for formulas in a suitable arithmetization of sintax[1].

Considering a different theory with respect to which the algorithm *decides* the deducibility, such a result can be improved in two ways:

1. strengthening (with respect to ZF) the theory when dealing with formulas that have been declared to be *not deducible* by the algorithm (we try to answer the question: for which theories F such that $F \supseteq ZF$ we can say

$$T(\varphi) \rightarrow (\neg D(\varphi) \rightarrow F \not\vdash \varphi)$$

2. weakening (with respect to ZF) the theory when dealing with formulas that have been declared to be *deducible* by the algorithm (here the question is: for which theories S such that $S \subseteq ZF$ we can say

$$T(\varphi) \rightarrow (D(\varphi) \rightarrow S \vdash \varphi)$$

Looking at the algorithm introduced in [1] it is easy to see that the formulas that have been declared to be *not deducible* with respect to ZF, as a matter of fact are not deducible in any consistent

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[1] We will use the term *deducibility algorithm* (or equivalently *deducibility problem*) which seems to be more appropriate in this context and avoids the possible misunderstandings that the term *decidability algorithm (decidability procedure)* could generate, since it would be always necessary to specify if we are talking about the decidability of *validity* or the decidability of *satisfiability*

extension of ZF, being the theory ZF *complete* with respect to those classes of formulas, therefore improvements of type 1. are trivial.

In this paper we present instead some improvement of type 2. by substituing the theory ZF with the weaker theory ZFA (where A stands for *absolute* or *arithmetic*) which is the theory ZF without the axioms of foundation, powerset and infinity presented in [5].

After developing the details for ZFA it will be clear how to further extend the results to even weaker set theories such as the theory T_0 presented in [2].

We will show that it is possible to solve the deducibility problem with respect to ZFA for the classes of formulas introduced in [1] by using the algorithm introduced in that paper modified by eliminating one of the steps, and we will prove the following metatheorem

> for any $\neg\varphi$ which does not contain an immediate contradiction of the axiom of foundation (a cycle of variables in the relation $\in$):
> $ZFA \vdash \varphi$ iff there exists a (consistent) extension F of ZFA such that $F \vdash \varphi$.

The method that we will describe is the one of the so called *permutations of the universe* of Fraenkel-Mostowski classically used to prove the consistency of the axiom of foundation and choice (for a complete introduction see [4]) and also used in [6] [7] [8] [9] to obtain a number of related results.

We will be able to use as metatheory again ZFA, which is strictly finitistic, since we will give consistency proofs by constructing internal syntactic interpretations and not by making use of the axiom of infinity to argue the existence of suitable models.

2. Decidability of a class of formulas.

Let us consider the language of set theory extended with the constant $\emptyset$ denoting the empty set. We define the class of formulas T_1

to be the propositional closure of the set of formulas of the type:

(2) $$x = y, \quad x \in y$$

where x and y are variables or the constant $\emptyset$.

We will call literals the formulas in T_1 of type (2) or their negations.

We will prove that there exists a deducibility algorithm for *disjunctions* of literals in T_1 with respect to ZFA. This will produce a deducibility algorithm for any formula in T_1 by first bringing generic formulas in T_1 in *conjunctive normal form*, and then testing each conjunct separately.

Deducibility algorithm for disjunction of literals in T_1.

Since we describe the algorithm in terms of satisfiability we start with a conjunction of literals φ. Let $\beta_1, \ldots, \beta_m$ be the literals in φ.

Step 1. Let V be the set of variables in φ; we define an equivalence relation $\sim$ on V in this way:

$x \sim y$ iff $x\rho y$, $x\rho x$ or x and y are the same variable;

where $x\rho y$ iff there exists a finite succession $x_1, \ldots, x_n$ of elements in V such that x is x_1, y is x_n and for any $j \in \{1, \ldots, n-1\}$ $x_j \hat{=} x_{j+1}$ (we will use the symbol ˆ on a connective of the form $=, \neq, \in, \notin$ to indicate that the corrensponding literal is in φ).

Let now $\tilde{V}$ be $V \backslash \sim$ (that is the set of equivalence classes in V with respect to the relation $\sim$).

We will indicate by $\tilde{\varphi}$ the formula obtained from φ by substituting the variables by their representatives with respect to the equivalence relation $\sim$ and eliminating all conjuncts of the form $x = x$, if the result is not empty; otherwise let $\tilde{\varphi}$ be any fixed tautology.

Observation 1. The existential closure of φ and $\tilde{\varphi}$ are logically equivalent (and therefore equisatisfiable).

Step 2. Consider the following conditions:

a) $x \hat{\neq} x$ for some $x \in V$.

b) $x \hat{\in} \emptyset$ for some $x \in V$.

c) $x \hat{\in} y$ and $x \hat{\notin} y$ for some $x, y \in V$.

and declare $\neg\varphi$ deducible iff one of the above condition is satisfied.

The claim expressed in step 2 is proved by showing the following lemma:

LEMMA 2.1. *None of the conditions a) b) c) is satisfied iff* $Con(ZFA + \exists\vec{x}\varphi)$[2]

Proof. Clearly if any of the conditions is satisfied then

$$ZFA \vdash \neg(\exists\vec{\varphi})$$

and therefore the theory $ZFA + \exists\vec{x}\varphi$ is not consistent.

Viceversa let us suppose that none of the conditions is satisfied and, moreover, let us suppose that also the following condition d) is not satisfied:

d) there exists $x_1, \ldots, x_n \in \tilde{V}$ such that

$$x_1 \hat{\in} x_2, \ldots, x_n \hat{\in} x_1$$

We will prove that if none of the conditions a) - d) is satisfied then $ZFA \vdash \exists\vec{x}\varphi$ from which it follows $Con(ZFA + \exists\vec{x}\varphi)$, and then we will see that the hypothesis d) is not necessary to prove the consistency.

From the fact that d) is not satisfied it easily follows that we can define a well order $\lhd$ on $\tilde{V}$ such that:

$$x \hat{\in} y \rightarrow x \lhd y$$

We will indicate with $x_1, \ldots, x_n$ the elements of $\tilde{V}$ assuming that $x_i \lhd y_i$ iff $i < j$.

2 Con stands for is *consistent*; $\exists\vec{x}\varphi$ stands for the existential closure of φ.

Let us suppose that $\emptyset \in \tilde{V}$ as a representative of the equivalence class to which it belongs or added by hypothesis if it does not appear in φ.

At any rate, because of the fact that condition b) is not satisfied, we can put $x_1 = \emptyset$.

Let I be a function having $\tilde{V}$ as domain with $Ix_1 = \emptyset$ and such that:

$$1 < j \leq n \rightarrow ((Ix_j = \{i_j\} \wedge i_j = \{j, n+1\}) \vee Ix_j = \emptyset).$$

We will see that considering $Ix_j = \emptyset$ only for $j = 1$ we already obtain the desired result but that a different choice of I will be necessary to obtain the generalizations presented in the following.

Because of the fact that condition d) is not satisfied we can say that $\hat{\in}$ induces a well founded relation on $\tilde{V}$ on which it is possible to define by recursion a function M as follows:

(3) $$Mx_j = Ix_j \cup \{Mx_i : x_i \hat{\in} x_j\}$$

We now show a lemma on the ground of which it will be possible to prove the previously stated result.

LEMMA 2.2.

a) for all $j, k \in \{1, \ldots, n\}$ $Mx_k \neq i_j$

b) if for all $j \in \{2, \ldots, n\}$ *we have* $Ix_j = \{i_j\}$ *then* M *is injective.*

Proof.

a) If $Ix_k = \{i_k\}$ then i_k is an element of Mx_k, but it cannot be an element of any i_j from the very definition of $\{i_j\}$, and therefore Mx_k is different from $\{i_j\}$. If, instead $Ix_k = \emptyset$ then either $Mx_k = \emptyset$ or $Mx_k = \{Mx_i : x_i \hat{\in} x_k\}$.
If it was $Mx_k = i_j$ we would have that for some x_i, $Mx_i = n+1$, but any Mx_i has at most n elements (this is because of (3) noticing that $|\tilde{V}| = n$) whereas $n+1$ has $n+1$ elements.

b) To prove that if for all $j \in \{2, \ldots, n\}$ $Ix_j = \{i_j\}$ then M is injective, it is enough to observe the definition of M and the fact that $j \neq k \rightarrow i_j \neq i_k$.

Now we show that if the function I is such that the assignment M is one-to-one then we have that:

$$\tilde{\varphi}_{x_1,\dots,x_n}(Mx_1,\dots,Mx_n)$$

is derivable in ZFA (that is the formula $\tilde{\varphi}$ with free variables $x_1,\dots,x_n$ is satisfied by the closed terms $Mx_1,\dots,Mx_n$ definable in ZFA).

We show that the terms Mx_i satisfy the literals in $\tilde{\varphi}$ from which the thesis will follow because of the definition of $\tilde{\varphi}_{x_1,\dots,x_n}(Mx_1,\dots,Mx_n)$.

If $x \hat{\neq} y$ then from the fact that a) is not satisfied we have that x and y are distinct and therefore $Mx \neq My$.

If $x \hat{\in} y$ from the definition of M it follows that $Mx \in My$ and therefore M satisfy $x \in y$.

If $x \hat{\notin} y$ then the from the fact that c) is not satisfied, it is not $x \hat{\in} y$. Moreover if $Mx \in My$ then $Mx = Mw$ with $w \hat{\in} y$ and since it is not $x \hat{\in} y$, it means that x and w are distinct whereas $Mx = Mw$: contradiction. Therefore M satisfies $x \hat{\notin} y$.

Now it will suffice to define $Ix_1 = \emptyset$ and $Ix_j = \{i_j\}$ for all $j \in \{2,\dots,n\}$ to obtain from Lemma 2.2 that M is one-to-one.

Observation 2. All the proofs presented above can be carried out in ZFA and therefore in case conditions a)-d) are not satisfied it follows that $ZFA \vdash \exists \vec{x} \varphi$.

Now we only need to prove that we can drop the hypothesis relative to condition d) still having $Con(ZFA \vdash \exists \vec{x} \varphi)$.

Let us suppose that $\tilde{\varphi}$ contains cycles and let us show how it is possible to build an internal representation in ZFA that satisfies the formula (sentence) $\exists \vec{x} \varphi$.

As we said in the introduction, we will use the method of the *permutations of the universe*.

Let us consider first the case in which the formula $\tilde{\varphi}$ contains only one cycle, then we will deal with the general case.

Let $\tilde{\varphi}'$ will be the formula obtained from $\tilde{\varphi}$ dropping one of the literals in the cycle, say $x \in y$.

The formula $\tilde{\varphi}'$ will not contain cycles and therefore it is satisfiable with a finite succesion of terms of the kind Mz.

Among the Mz's we must find also Mx and My (because of the fact that in the cycle there were two literals of the form $y \in v$ and $u \in x$).

The succesion of Mz's will satisfy, in particular, the cycle of $\tilde{\varphi}$ deprived of the literal $x \in y$, that is:

$$My \in Mv \wedge \ldots \wedge Mu \in Mx$$

Let us consider a functional bijective relation F (permutation of the universe) that *swaps* the sets My and $My \cup \{Mx\}$ and does not touch any other set.

Observation 3. For any variable z in $\tilde{\varphi}$ different from y we have:

$$My \cup \{Mx\} \neq Mz$$

because $i_y \in My \cup \{Mx\}$ and $i_z \in Mz$ with $i_j \neq i_z$ (apply lemma 2.2).

Let ε be the following relation:

$$a \varepsilon b \leftrightarrow a \in F(b) \tag{4}$$

Let us consider the universe of sets $\mathcal{V}$ as domain and ε as interpretation of the membership relation.

We have that the structure $\langle \mathcal{V}, \varepsilon \rangle$ is an internal interpretation for ZFA (see [4] chapter 7) in which $\exists \vec{x} \varphi$ is satisfied.

In fact: if in (4) for b there is an element for which F is the identity then the relation ε coincides with $\in$; therefore let us analize only the case in which b is My or $My \cup \{Mx\}$.

Notice that from observation 3 we can conclude that $My \cup \{Mx\}$ does not appear among the terms satisfying $\tilde{\varphi}'$ and therefore we are only interested in the case in which b is My.

Because of the fact that $My \subseteq F(My)$ we can say that if $Mz \in My$ then $Mz\varepsilon My$; moreover, from the fact that $Mx \in F(My)$ we can also conclude that $Mx\varepsilon My$, that is the assignment M satisfies $x \in y$ with respect to the relation ε.

If $z\hat{\notin}y$ then $Mz\notin M_y$ from which $Mz\not\varepsilon My$ because the only element that is in relation with My with respect to ε and does not belong to My is Mx and if $Mz = Mx$ then $z = x$, but we supposed $z\hat{\notin}y$ and therefore we would have $x\hat{\notin}y$, we also have $x\hat{\in}y$ and this is a contraddiction because c) is not satisfied.

The observation 3 solves the case of literals of the form $z{\neq}y$ and therefore the assignment M satisfies, with respect to ε, all the literals that it satisfies with respect to $\in$; moreover it satisfies also $x \in y$ and therefore it satisfies $\tilde{\varphi}$.

Observation 4. The proof now presented is valid also in the case in which the variables x and y coincide. It will suffice to define $Mx = \{i_x\}$ if x does not appear in any other literal in $\tilde{\varphi}$ and then proceed with the very same argument.

The case in which the formula contains more than one cycle is solved by induction on the number of cycles:

if $\tilde{\varphi}$ contains only one cycle we will repeat the previous proof, if $\tilde{\varphi}$ contains n cycles let us eliminate one literal per cycle and let us suppose that we already defined the relation ε_{n-1} with respect to which the first $n-1$ cycles are satisfied.

Let $x \in y$ be the literal eliminated from the n-th cycle, let us define ε_n as follows:

$$a\varepsilon_n b \leftrightarrow a\varepsilon_{n-1} F(b)$$

where F permutates M_y with $My \cup \{M_x\}$.

Letting ε_n and ε_{n-1} play the roles of ε and $\in$ respectively in the previous proof, it is easy to check that the terms of the form Mx

satisfy the formula $\tilde{\varphi}$ with respect to the relation ε.

We have showed that for any formula φ, conditions a) b) c) do not hold iff $Con(ZFA + \exists\vec{x}\varphi)$.

Moreover we have that:

$$Con(ZFA + \exists\vec{x}\varphi) \leftrightarrow ZFA \nvdash \forall\vec{x}\neg\varphi$$

and therefore, recalling that the negation of a conjunction of literals is a disjunction of literals, we have a decision algorithm for disjunction of literals in the class of formulas T_1.

Let us note now that all we did till now is perfectly adaptable to the theory ZF, it will suffice to add to conditions a) b) c) of lemma 2.1 the condition d) that will have the only effect of declaring deducible negations of formulas containing immediate contradictions to the axiom of foundation (i.e. cycles).

This observation allow us to conclude that not only the class T_1 is decidable with respect to ZF but, moreover, that ZF is complete with respect to T_1; this fact alone implies the decidability; notice that in general the viceversa does not hold and T_1 with respect to ZFA is just an example of this fact.

Let us now prove the completeness of ZF with respect to T_1: if φ a disjunction of literals and if $\neg\varphi$ satisfies one among a)-c) then it follows that $(ZFA + \exists\vec{x}\neg\varphi)$ is not consistent, from which $ZF \vdash \forall\vec{x}\varphi$; on the hand, if $\neg\varphi$ satisfies d), then it follows from the axion of foundation that $ZF \vdash \forall\vec{x}\varphi$. Finally, if $\neg\varphi$ does not satisfy any of the four previous conditions, by an argument similar to the proof of Lemma 2.1 it follows that these hypothesis $ZF \vdash \exists\vec{x}\neg\varphi$ and therefore $ZF \vdash \neg(\forall\vec{x}\varphi)$.

Let us conclude by giving a proof of the metatheorem stated in the introduction.

The only if part is trivial; the converse let us suppose that $ZFA \nvdash \varphi$, that is $Con(ZFA + \exists\vec{x}\neg\varphi)$ and therefore $\neg\varphi$ does not sotisfy neither a) nor b) nor c), moreover $\neg\varphi$ does not satisfy d) from the hypothesis and hence $ZFA \vdash \exists\vec{x}\neg\varphi$ from which $F \vdash \exists\vec{x}\neg\varphi$ which implies $F \nvdash \varphi$ from the consistency of F.

3. A satisfiability algorithm D

Syntactivally we say that a formula is satisfiable with respect to a certain theory P if P together with the existential closure of φ is consistent, that is if $Con(P + \exists\vec{x}\varphi)$.

Let us note that $Con(P + \exists\vec{x}\varphi)$ iff $P \nvdash \neg(\exists\vec{x}\varphi)$ iff $P \nvdash (\forall\vec{x}\neg\varphi)$ and therfore we can conclude that if C is a class of formulas such that

$$\varphi \in C \leftrightarrow \neg\varphi \in C$$

then the problems of satisfiability and deducibility for C are equivalent.

In this section we will introduce an algorithm that will allow us to solve (in the next section) the satisfiability problem for a particular class of quantified formulas that has the previous property and hence we will be able to say that for this class, the deducibility problem is solved.

We will say *sound* an algorithm that transforms a satisfiable formula φ in a satisfiable formula φ'; we will say *complete* an algorithm that is such that if φ' is satisfiable then also φ is satisfiable.

Let T be a class of unquantified formulas closed with respect to the propositional connectives.

We will call *prenex-T-formula* any formula of the form:

$$Q_1, \ldots, Q_n R$$

where all the Q_i's are equal to $\forall x_i$ or $\exists x_i$ and $R \in T$.

Let Σ_T be the propositional closure of the set of prenex T-formulas.

Let Σ'_T be the set of T-formulas of Σ_T of the form:

$$\beta_1 \wedge \ldots \wedge \beta_n$$

where all the β_i's are prenex T-formulas.

Clearly the satisfiability problem for Σ_T is solved if it is solved the satisfiability problem for Σ'_T and, moreover, we can suppose that in any prenex T-formula two different quantifiers will bound two different variables and that no variable will occur both free and bounded (possibly renaming the variable and passing to a logically equivalent formula).

The algorithm D that we are now going to describe will transform formulas φ in Σ'_T into formulas in T; we will see that it is sound but, in general, not complete (we will prove the completeness in a particular case presented in the next section) and for any $\varphi \in \Sigma'_T$ it will answer to the question if φ satisfiable or not with respect to a theory for which T is decidable.

Let us initialize V to the set of free variables in φ (if $V = \emptyset$ we put $V = \{z\}$ for a new variable z).

Notice: V will be modified during the procedure by adding the variables that will then result in the formula we are going to obtain.

Step 1. Substitute the conjuncts of the form $\exists x\psi(x)$ with $\psi(x)^x_{z_x}$ (that is the formula obtained substituing all the occourrencies of x by z_x) with z_x new variable that we add to V.

Repeat the step as many times as possible.

Step 2. Let $\bar{\varphi}$ be the formula obtained after all the applications of the previous step; if $\bar{\varphi}$ belongs to T then apply the decision algorithm for T and if $Con(ZFA + \exists\vec{y}\varphi)$ we can say that $Con(ZFA + \exists\vec{x}\varphi)$, if otherwise $\neg Con(ZFA + \exists\vec{x}\bar{\varphi})$ the algorithm will answer *I do not know* (this is the case in which the algorithm reveals its incompleteness).

Step 3. If $\bar{\varphi} \notin T$ consider a disjunct fo the form $(\forall x)\psi(x)$ and substitute it with $\wedge_{w \in V} \psi^x_w$.

Go back to step 1.

Clearly Step 1 is both sound and complete whereas it is easy to verify that Step 3 is still sound but, in general, not complete (see [1] for a complete discussion).

We now introduce the notion of *Special Set of Variables* that we will use in the next paragraph to show that in some particular case the algorithm D is complete.

DEFINITION 3.1. *A set S of variables is said to be special if the following conditions hold:*

1. $\emptyset \in S$;
2. *for all* $x, y \in S$ *there exists a* $z \in \tilde{V}$ *such that* $(z\hat{\in}x \leftrightarrow \neg z\hat{\in}y)$

Observation 1. Strictly speaking the previous definition should specify also the formula with respect to which S is said to be special, but this will always be clear from the context.

LEMMA 3.2. *if φ does not satisfy conditions a)-d) of lemma 2.1, the following facts are equivalent:*

A) *the function I is defined in such a way that if x and y are distinct then $Mx \neq My$.*

B) *the set $S = \{x : Ix = \emptyset\}$ is special.*

Observation 2. $S = \{\emptyset\}$ is special.

Proof. (of the lemma 3.2) If A) holds then $I\emptyset = \emptyset$ by definition and therefore $\emptyset \in S$.

Moreover if x and y are distinct in S but, by contradiction, such that

$$z\hat{\in}x \leftrightarrow z\hat{\in}y$$

then from the fact that they are in S we can conclude that $Ix = Iy = \emptyset$ and moreover

$$\{Mz : z\hat{\in}x\} = \{Mz : z\hat{\in}y\}$$

that is $Mx = My$ which is a contradiction and therefore B) holds.

If B) holds let $h(x)$ be the number associated to x in the enumeration of $\tilde{V}$ and let us call $h(x)$ the *height* of the variable x.

Let us prove A) by induction on $k = \max(h(x), h(y))$.

If $k = 0$ then x and y cannot be distinct and therefore there is nothing to prove.

Let now A) be true for $m < k$ with $(k \geq 1)$.

If $x, y \in S$ then for some z it must be that (for example) $z \hat{\in} x$ and $\neg z \hat{\in} y$, from which it follows that $Mz \in Mx$.

If, by contradiction, $Mz \in My$ then $Mz = Mw$ for some w such that $w \hat{\in} y$ and because of the fact that it is not $z \hat{\in} y$ we have that w and z are distinct and by inductive hypothesis $Mw \neq Mz$: contradiction.

Therefore $Mz \notin My$ from which $Mx \neq My$.

If $x, y \notin S$ then $Ix \neq Iy$ and therefore $Mx \neq My$.

COROLLARY 3.3. *Let S be a special set of variables.*

If I is the function such that $Ix_j = \emptyset$ when $x_j \in S$ whereas $Ix_j = \{i_j\}$ otherwise, then the corresponding M satisfies $\exists \vec{x} \varphi$ whenever φ does not satisfy conditions a)-d) of lemma 2.1.

Observation 3. (important) if $x \in S$ and S is special, then $Mx = \{Mz : z \hat{\in} x\}$ (that is every element of Mx is a Mz with z variable of φ).

4. A case in which the algorithm D is complete.

Now we define a class of prenex formulas for which the algorithm D is not only sound but also complete.

DEFINITION 4.1. *a T_1-formula q is said to be simple iff q is of the form*

$$Q_1, \ldots, Q_n P$$

and moreover:

a) every Q_i is of the type

$$(\forall y_i \in z_i)$$

or every Q_i is of the type

$$(\exists y_i \in z_i)$$

and P is a formula in T_1.

b) no y_i is a z_i (that is every z_i is free).

Let Γ_1 be the proposition closure of the class of simple prenex T_1-formulas.

From now on let V_χ indicate the set of free variables in the formula χ.

Let also Φ be the formula before applying the algorithm and φ be the corresponding formula after applying the algorithm D. We will suppose φ to be satisfiable since we want to prove completeness.

As usual, given the formula in Γ_1 that we want to test, we will bring the formula in disjunctive normal form (let us note that the negation of a simple prenex T_1-formula is still a simple prenex T_1-formula) and we wil test each one of the disjuncts, which will result to be conjunctions of simple prenex T_1-formulas.

Hence it is not restrictive to describe the algorithm (and to show its completeness) in the case in which Φ is of the form

$$\beta_1 \wedge \ldots \wedge \beta_k$$

where for all i, β_i is a simple prenex T_1-formula.

Let us recall that the only step of the algorithm which could create problems was step 3 (the others were sound and complete).

First of all let us consider the following example that will show why it is necessary to apply a *normalization procedure* to the formulas to which we want to apply the algorithm.

Let Φ be the formula:

$$(\forall x \in y)(x \notin y) \wedge (\forall w \in z)(w \notin z) \wedge y \neq z.$$

The formula is clearly unsatisfiable, but if we apply D we obtain the following φ:

$$(z \in y \rightarrow z \notin y) \wedge (y \in y \rightarrow y \notin y) \wedge$$

$$\wedge (y \in z \rightarrow y \notin z) \wedge (z \in z \rightarrow z \notin z) \wedge y \neq z$$

which is equivalent to:

$$(z \notin y) \wedge (y \notin y) \wedge (y \notin z) \wedge (z \notin z) \wedge z \neq y$$

which, in turn, is clearly satisfiable.

In this case we have that φ is satisfiable whereas Φ is not and therefore the algorithm D is not complete.

The necessity of applying the normalization procedure *before* the algorithm D will imply that, even if we will say that D is complete, actually the procedure D' consisting of the normalization procedure followed by the algorithm D will be complete with respect to Γ_1.

Normalization Procedure: for every pair of distinct variables x, y in V_Φ add the following conjunct:

$$(x = y \vee (z_{x,y} \in x \wedge z_{x,y} \notin y) \vee (z_{x,y} \notin x \wedge z_{x,y} \in y))$$

where $z_{x,y}$ is a new variable.

Clearly the normalization procedure transforms formulas in formulas equisatisfiable; moreover let us suppose that the formula φ turns out to be satisfiable.

The normalization procedure will allow us to conclude tha the set of variables V_Φ is special for φ and therefore we can define an assignment I such that

$$Ix = \emptyset \leftrightarrow x \in V_\Phi$$

Using the assignment I we can define an interpretation M such that:

(*) for all $x \in V_\Phi$ if Mx has only elements of the form Mz for some $z \in V_\varphi$

Let us suppose that φ does contain cycles. In this case we can apply the procedure described in section 2 by eliminating a literal from every cycle but making sure never to eliminate a literal of the form:

$$z_{x,y} \in x$$

with $x \in V_\Phi$.

At this point we can conclude that V_Φ is special for the formula φ' obtained from φ after *breaking* all the cycles.

Using the interpretation M which satisfied (*) and defined for φ' we can then define the relations ε_n as we did in section 2.

Let us note that for every n, if for some x the elements for which relation ε_{n-1} holds between them and Mx are all of the form Mz, then the same is true also for the same x and the relation ε_n, and therefore (*) continues to hold.

At this point it is clear that if $z_i \in V_\Phi$ and if

$$\bigwedge_{h \in V_\phi} (h \in z_i \rightarrow \varphi_h^{y_i}(h))$$

holds, then also

$$(\forall y_i \in z_i)\varphi(y_i)$$

holds, and therefore Step 3 is complete.

We can finally observe that if the class T_1 from which we started was *complete* with respect to the underlying set theory, then the previous algorithm shows the completness of the class Γ_1, and this is the case for the theory ZF.

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